Simple harmonic oscillations and damped oscillations

AC ocillation circuit: resistance, coil and capacitor in series.

Kirchoff's law: $U_L + U_R + U_C = 0$

<=> L I ' + R I + Q/C = 0

<=> L Q" + R Q' + $\frac{1}{C}$Q =

<=> Q " + $\frac{R}{L}$Q ' + $\frac{1}{LC}$Q = 0

This is called a second degree, linear , homogenous differential equation with constant coefficients.
All circuits can be described by this kind of equations.
To get a unique solution, we have to know two initial conditions, which determine the integration constants.

Theory

1. For equation y " + b y' + c y = 0 the solutions are of form e^{rt}

2. Substitution gives $r^2 e^{rt}$ + b r e^{rt} + c e^{rt} = $(r^2 + b r + c)\, e^{\mathrm{rt}} = 0$

3. This is possible only for r satisfying *characteristic polynomial*
   r^2 + b r + c = 0
4. This polynomial has two roots

Case 1: If b = 0 and c > 0, roots are r = $\pm i\sqrt{c}$. Using $\omega = \sqrt{c}$
solution: y = A $e^{i\omega t}$ + B $e^{-i\omega t}$ or using trig. form C sin(ω t + φ)

Case 2: If b ≠ 0 , we have two roots r = $-\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$

When $b^2 > 4$c , there is no imaginary part => there is no oscillations, just "damping"

solution: y = A $e^{r_1 t}$ + B $e^{r_2 t}$

When $b^2 < 4$c , there is real part and imaginary part, there is a damping and oscillation

solution: y = A $e^{r_1 t}$ + B $e^{r_2 t}$ = $e^{-\frac{b}{2}} (C_1 e^{i\omega t} + C_2 e^{-i\omega t})$ = $e^{-\frac{b}{2}}$ (C sin(ω t + φ),

where oscillation $\omega = \sqrt{c - b^2/4}$

LRC oscillation circuit

1) If resistance R = 0, we have simple harmonic oscillation $Q'' + \frac{1}{LC}Q = 0$.

Solution is $Q = Q_0 \sin(\omega t + \varphi)$,

where $\omega = \frac{1}{\sqrt{LC}}$ and frequency $f = \frac{1}{2\pi\sqrt{LC}}$, φ = phase

In exponential form $Q = Q_0 \; e^{j(\omega t + \varphi)}$

2) If resistance R > 0, we have simple harmonic oscillation $Q'' + \frac{R}{L}Q' + \frac{1}{LC}Q = 0$.

Solution is $Q = Q_0 \, e^{-\frac{R}{2L}t} \sin(\omega t + \varphi)$, where $\omega = \frac{1}{\sqrt{LC - \frac{R^2}{4L^2}}}$ and , φ = phase

Notice: Damping resistance R seems to change also the oscillation frequency.

Amplitude decreases according to $e^{-\frac{R}{2L}t}$

Animation

In this animation parameter c = 10000. So we expect that ω = 100, f = 15.92 and T = 0.063 sec
Damping parameter b is changed from 0 to 500.

One would expect that when $\frac{b^2}{4}$ = c = 10000 , which means that b = 200, the oscillations vanish.

Already before that oscillation frequency tends to go down and T tends to increase towards infinity.

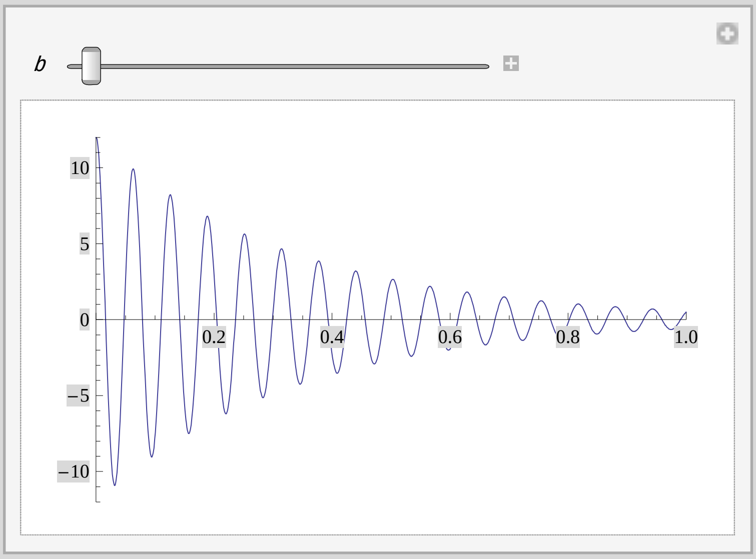
In[7]:=
```
mySHM[b_] := Module[{},
  solu = DSolve[{y''[t] + b y'[t] + 10^4 y[t] == 0, y[0] == 12, y'[0] == 0}, y[t], t];
  Plot[Re[y[t] /. solu] , {t, 0, 1}, PlotRange → {{0, 1}, {-12, 12}}]
 ]
```

Solutions of y'' + b y + c y = 0 , where c = 10000
In the model below the value of damping parameter b is changed dynamically.

In[8]:=

```
Manipulate[mySHM[b], {b, 0, 500}]
```

Out[8]=



In the next dynamic model , we change be and follow the change of the roots of the **characteristic polynomial**.
If roots are imaginary, then ω = the imaginary part of the roots and the coefficient α in the damping factor $e^{-\alpha t}$ is the real part of the roots

In[1]:=

```
(*  Roots of characteristic polynomial r^2+ b r + c = 0 *)
c = 10 000;
Manipulate[{-b/2 - Sqrt[b^2 - 4 c]/2, -b/2 + Sqrt[b^2 - 4 c]/2} // Chop, {b, 0, 500}]
```

$$\text{Manipulate}\left[\left\{-\frac{b}{2}-\frac{\sqrt{b^2-4c}}{2},\ -\frac{b}{2}+\frac{\sqrt{b^2-4c}}{2}\right\} \text{// Chop},\ \{b, 0, 500\}\right]$$

Out[2]=

b

{−100. *i*, 100. *i*}

Examples:

b = 0	$y = A \sin(100\, t + \varphi)$	ω = 100 rad/s
b = 51	$y = A\, e^{-25.5 t} \sin(96.7\, t + \varphi)$	ω = 96.7 rad/s
b = 151	$y = A\, e^{-75.5 t} \sin(65.6\, t + \varphi)$	ω = 65.6 rad/s
b = 200	$y = A\, e^{-200 t} + B\, e^{-50 t}$	no oscillation anymore

Summary

As we saw, the behavior of a system depends only on the roots of the characteristic polynomial.

1) 2 purely imaginary roots $\pm i\,\omega$ <=> simple harmonic oscillations $y = A \cos(\omega t + \varphi)$
2) 2 complex roots with real parts $\alpha \pm i\,\omega$ <=> damped oscillations $y = A\, e^{-\alpha t} \cos(\omega\, t + \varphi)$
3) real double root r1 = r2 = r < 0 <=> critically damped system $y = A\, e^{r t} + B\, t\, e^{r t}$
4) 2 distinct negative real roots r1and r2 <=> over damped system $y = A\, e^{r1\, t} + B\, e^{r2\, t}$

Example: Solve y'' + 2y + 4 = 0

Solve the root of characteristic polynomial

In[6]:=
```
r^2 + 2 r + 4 == 0.  // Solve
```

Out[6]=
$\{\{r \to -1. - 1.73205\, i\}, \{r \to -1. + 1.73205\, i\}\}$

Solution is $y = A\, e^{-t} \cos(1.732\, t + \varphi)$.

Amplitude A and phase φ are determined using some initial condition , like knowing y(0) and y'(0)